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New Product Introductions: Improving Demand Information and Supply Responsiveness

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This study addresses the challenge of matching demand and supply in the product introduction phase. We model the effects of prepositioning strategies designed to increase responsiveness and adopt a dynamic approach which permits linking to management of products that have reached steady state. Empirical calibration and evaluation of this strategy is made possible using data from a major cosmetics company marketing their own products in the Middle East and where the approach has been implemented.

Key words: New product introduction, Bayesian inventory, Real options

History: Working paper

1. Introduction

In today's global market place, components of a product are often manufactured in different geographical locations before all parts are shipped and assembled into a finished unit. This of course increases the complexity of the logistics of inventory management, but first and foremost it makes lead times significantly longer. Inventory decisions have become all the more risky. This is particularly the case when launching a new product, because supply commitment has to be done at a time when forecast accuracy greatly suffers from lack of actual historical demand data. This study is concerned with addressing the challenge of matching demand and supply in the product introduction phase in this global sourcing context. One of its key contributions consists of a dynamic approach which permits linking the introduction phase to management of product's stable phase, often referred to as steady state in inventory models. It also contributes to the field of product life cycle management, where uncertainty is traditionally not in focus.

In the traditional approach of initial order is to order only finished products. This will constitute our base model as well as our benchmark. However, more often than not, demand uncertainty is greatly reduced, if not virtually lifted, very soon after product launch, when initial sales have been observed and a more accurate forecast can be made. This is particularly true for fashionable products, where the level of uncertainty typically plunges from very high levels as soon as the product is on the market, and success is obvious, or failure evident. This can be thought of as the evolution of demand uncertainty, which is reduced as demand information in the form of actual sales becomes available. The supply process also follows an evolution where costs are gradually incurred from the initiation of component acquisition to the time where the products have been completed and transported with associated customs clearance. The two forms of evolution can be utilized in order to device a component pre-positioning strategy. Here, in addition to ordering some quantity of finished products ahead of the product launch, the firm orders additional semi-finished products in the form of components. Hence, an option for short lead time is constructed at only a part of the unit cost commitment of a finished product. This has two main benefits: First, in case the product is a hit, the pre-positioned quantity can be quickly turned into finished products to reduce the extent of lost sales. Second, in case the product is a dog, the total exposition in inventory investment might be less thus reducing overage costs and reducing the time necessary to reach “steady state”.

In our model, we consider terminal leftover inventory at the end of the introduction phase in order to take into account necessary time to reach steady state and an achievable optimal order policy, as well as its impact on cost. Analytically tractable solutions are obtained for both cases and we determine optimality conditions for finished units and component order quantities as well as for the number of components to assemble after observing demand.

For better matching demand and supply, in addition to improving supply responsiveness until better demand information is available, one can obtain advance demand information prior to supply commitment. In an effort to address this second approach, we explore different ways to improve the quality of the demand distribution prior. We empirically calibrate the characteristics of the actual distribution, and investigate how to best improve forecasts, such as improved initial forecast based on subjective estimates and through strategies such as pre-order test sales which results in actual sales data prior to the commitment of the initial regular order.

Our models have been used to improve the management of new product introductions at a major cosmetics retailer in the Middle East, which has its own brand and designs. As an illustration, we give an example product of which the model was used for. Of the total lead time of 7.5 months, only 5.65 AED out of the total cost of 20.11 AED is incurred during its first 5.5 months. With unit revenue of 59, with the base model expected operating profit in the introduction phase of 8 months (with adjustments for impact on stable phase) is $84,914$ based on order quantity of $4,440$ finished units. Using the pre-positioning strategy, the model suggest ordering $2,987$ finished units and $2,861$ component sets resulting in expected operating profit of $94,100$. This is an 11% increase over the base model, and represents 6% of revenues. This paper reports on the implementation of these models at the firm. Using econometrics analysis, we estimate its impact on Key Performance Indicators such as lost sales, excess inventory and operating profit.

2. Literature Review

a. Other papers dealing with introduction phase b. Newsvendor with timing flexibility and Bayesian updating

New product introduction and other papers that consider similar strategies. People look at different things in isolation

- Timing flexibility

- Product flexibility

3. Context and Model Formulation

The main goal of this paper is exploration of inventory decisions for products during early stages of product life cycle. We consider this problem for products which have medium-to-long life cycle and have in general more than one replenishment opportunity, i.e., they do not fit in style goods category. Since this research is motivated by a real-life business situation, it befits to provide a brief description of the context for this research along with model formulation. We consider a firm that regularly launches new product (once a year) and sells these products through their own retail outlets, i.e., they face the consumer demand directly have up-to-date information on the demand. Prior to the launch and during the initial stages of product life-cycle, the demand estimates are usually derived from the intuition managers have about the market acceptance of the product, are rather imprecise. The firm therefore faces high level of uncertainty in product demand during this stage. After the product launch, product's sales for first few weeks serve as a good indicator of products acceptance in the market. The firm uses this information to arrive at more precise longer term demand estimate for the product. Thereafter the firm manages inventory of the product based on this improved estimate, until the end of product's life cycle.

We model demand as a time continuous stochastic process with unknown parameter. We denote the demand incurred in time interval $(t_1, t_2]$ by $D_{(t_1,t_2]}$. $\mathbb{I}_t$ denotes information available at time t about the demand process (including potential learning), $\mathbb{E}_t$ and $\mathbb{P}_t$ denote the expectation and probability operator, respectively, conditional on $\mathbb{I}_t$. The demand and inventory holding cost is incurred continuously in time and are accounted for as such.

The product is supplied by an external supplier, who quotes lead-time L and unit cost C for supplying product units. A more detailed scrutiny of procurement process reveals that it comprises of two stages: In the first "sourcing stage", components are procured from suppliers, and in the second "assembly stage" components are assembled together into finished products and shipped to stores. A product unit may require multiple components, and in such cases we focus on component set (i.e., a set of components that make one unit of finished product). For simplicity, throughout this paper we use the term "component" to refer to "component set" and term "product" to refer to "finished product". We denote the lead-time for the sourcing stage by l_1 and associated unit component cost by c_1; assembly stage lead-time is denoted by l_2 and unit assembly cost by c_2. The complete procurement of the product thus has total lead time $L = l_1 + l_2$ and has total unit cost $C = c_1 + c_2$. We use notation R to denote the retail price for a product unit. In addition to cost of procurement, each unit of product and component incur holding cost when they are held in inventory. The holding cost rate of finished product is denoted by H and that of component is denoted by h.

The timeline of decisions made by the firm for a product is as follows: Let $t = 0$ denote the time at which the product is launched. The first inventory decision is made at time $t = -L$ to ensure availability of product at the launch. After the product launch the firm revises the demand information based on actual observed sales, and makes subsequent inventory decisions with better information. Let $t = t_o$ be the time by which most of the uncertainty in the market acceptance of the product is resolved. We refer to the period from $t = 0$ to $t = t_o$ as the observation period. An order placed after observation period can affect product inventory levels no earlier than a full lead time after the end of observation period, i.e., at time $t = t_o + L$. In other words, the firm can respond to the level of market acceptance and achieve stable state inventory levels no earlier than time $t = t_o + L$. We therefore define time interval $(0, t_o + L]$ as the introduction phase and focus on decisions that impact inventory during this period.

The issue of deciding order quantity while facing random demand is adequately captured by a simple Newsvendor model. However, in the context considered in this paper, several other challenges are present which are not addressed by the Newsvendor model. The first of these challenges is to update demand information after observing initial sales, and respond to this updated information. In order to address this issue, we propose "component pre-positioning strategy", which exploits the two stage nature of the procurement process to build quick response capability without additional cost. With this strategy, the firm places order in two stages: At time $t = -L$, it orders finished product units as well as components. Finished product units are shipped directly to the firm and made available for product launch at time $t = 0$, whereas components are pre-positioned to be assembled later. At time $t = t_o$, after observing initial sales, the firms placed additional order to assemble and ship pre-positioned components, thus achieving quick response to improvement in demand information. In order to incorporate learning about demand, we consider the a model of demand process that allows updating demand distribution upon observing initial sales. We envision the total uncertainty in a product's demand to comprise of two types of uncertainties: (i) systemic uncertainty in demand due to fluctuations in market, (ii) product specific uncertainty due to the fact that the level of market acceptance for a product is difficult to ascertain prior to launch. We refer to the former as market uncertainty, and the latter as product uncertainty. In the online supplement, we specify how this structure can be incorporated for different demand distributions.

In many cases, some components are not exclusive for a particular SKU (Stock Keeping Unit), but rather shared between multiple SKUs of similar characteristics. For example, 8 SKUs belonging to a line of lipstick carried by the firm have a common "housing", while they differ in their shades, which is determined by the "stick" for each SKU. In this example, housing is a common component and stick is an SKU specific component. Note that the component specificity becomes important only if the lead-time for component is long, in which case multiple types of components have to be ordered in advance, resulting in higher potential for mis-match between demand and supply. If the specific component can be bought at short lead time, the SKU mix decision can be made closer to the time of assembly and the cost of the specific component can be folded in cost of assembly. In other words, if its lead time is reasonably short, an SKU specific component can be treated as a common component. When combined with component pre-positioning strategy, commonality of components allow the firm to postpone to a part of product mix decision until the end of observation period (this flexibility is in addition to flexibility in finished product inventory decision allowed by pre-positioning strategy). However this flexibility is compromised due to Minimum Order Quantity (MOQ) the supplier requires when accepting orders.

The third major challenge in the current context is in terms of linking inventory decisions in the introduction phase to management of inventory during the stable phase. As long as it is achievable, the product will reach the first steady-state inventory cycle by $t = t_o + L$. In case of large excess leftover inventory at $t = t_o + L$ from the introduction phase, the first steady-state inventory cycle cannot be achieved until inventory depletes sufficiently. In other words, the firm would end up carrying larger than optimal inventory during some portion of the stable phase. Therefore, the cost of excess inventory during the introduction phase is not only the cost of holding inventory, but also the additional inventory cost incurred during the stable phase due to leftover inventory at $t = t_o + L$. This additional cost of excess inventory is a function of leftover inventory at the end of introduction phase as well as the market acceptance level for the product (since the optimal steady-state inventory depends on it). We define expected terminal cost function $V(x, y, \mathbb{I}_t)$ to capture estimate of this cost at time t, where x and y, respectively, are product and component leftover inventory at $t = t_o + L$, and $\mathbb{I}_t$, the information available at time t about the market acceptance of the product. We assume that $V(x, y, \mathbb{I}_t)$ is jointly convex in x and y. In §**??**, where we illustrate a functional form of $V(x, y, \mathbb{I}_t)$, we verify this assumption.

Base Case Decision Making First consider the base case model of the introduction phase (i.e., a status-quo model without additional strategies). We will use this model as benchmark to evaluate strategies analyzed subsequently. The impact of initial order quantity Q on expected profit earned during introduction phase is,

$$\begin{aligned}\Pi^b(Q) = {} & R\mathbb{E}_0 \min\left\{D_{(0,t_o+L]}, Q\right\} - CQ - H\int_0^{t_o+L} \mathbb{E}_0\left(Q - D_{(0,t]}\right)^+ dt \\ & -V\left(\left(Q - D_{(0,t_o+L]}\right)^+, 0, \mathbb{I}_{t_o}\right),\end{aligned} \tag{1}$$

where the first term is revenue earned through sales in time period $(0, t_o + L]$, second term is cost for procuring Q units, the third term is cost of holding inventory and the last term is the expected cost of left-over inventory at the end of introduction phase. The function $V_1(x, \mathbb{I}_{t_o})$ characterizes the expected cost of left-over inventory x at the end of the introduction phase. Since, the desired steady-state inventory level after the introduction phase depends on the market acceptance level for the product, the cost of left over inventory is dependent on this realized information $\mathbb{I}_{t_o}$ at time $t = t_o$. If $V_1\left(\left(Q - D_{(0,t_o+L]}\right)^+, \mathbb{I}_{t_o}\right)$ is convex in Q, then the expected profit is maximized at Q^*, which is the unique solution to the following first-order condition:

$$\mathbb{P}\left(D_{(0,t_o+L]} \le Q\right) + \frac{H}{R}\int_0^{t_o+L} \mathbb{P}\left(D_{(0,t]} \le Q\right) dt = \frac{R - C - \dfrac{\partial}{\partial Q} V\left(\left(Q - D_{(0,t_o+L]}\right)^+, 0, \mathbb{I}_{t_o}\right)}{R}. \tag{2}$$

4. Illustration of Specific Issues

In this section, we delve deeper into the specific aspects of the problems discussed in the previous section. Since these issues are sufficiently complex, analysis of a model that incorporates them together is prohibitive. Therefore, for each case we consider simpler model that highlights the specific aspect being focused on.

4.1. Updating Demand and Responding to Updates: Pre-Positioning Strategy

In order to illustrate the value of pre-positioning strategy along with updating demand using Bayesian framework, we consider a simplified version of our original problem. More specifically, throughout this section, we assume the following:

ASSUMPTION 1. *(a)* $H = h = 0$. *(b)* $\mathbb{P}_0\big(D_{(0,t_o]} > Q\big) \approx 0$. *(c)* $V(x, y, \cdot) = S\mathbf{1}_{\{x>0\}} + s\mathbf{1}_{\{y>0\}}$.

Assumptions $1(a)$ is reasonable when cost of holding inventory is very small relative to the unit margin on product units. Assumptions $1(b)$ is a reasonable as long as t_o is sufficiently small and theboptimal values of Q sufficiently large to ensure that the probability of stockout during the observation phase is negligible. Assumption $1(c)$, suggests salvaging leftover inventory after the introduction phase, and hence undervalues this inventory as these units can be used towards satisfying demand incurred after introduction phase. Throughout this section, for notational simplicity we denote $D_0 \equiv D_{(0,t_o]}$, $D_1 \equiv D_{(t_o,t_o+l]}$ and $D_2 \equiv D_{(t_o+l,t_o+L]}$.

Applying Assumptions $1(a)$-(c) on the base case of our model without pre-positioning strategy, we obtain the following expression for expected profit

$$\Pi^b(Q) = R\mathbb{E}_0 \min\{D_0 + D_1 + D_2, Q\} + S\mathbb{E}_0(Q - D_0 - D_1 - D_2)^+ - CQ. \tag{3}$$

The optimal order quantity Q^b is solution Q to the following Newsvendor optimality condition

$$\mathbb{P}_0(D_0 + D_1 + D_2 \leq Q) = \frac{R - C}{R - S}. \tag{4}$$

Now consider pre-positioning strategy under these assumptions. The optimal quantity decision involves two steps. First is finding optimal number of units q_2 to assemble at time t_0, given initial decisions Q and q_1 and demand observation $D_0 = d_0$. The objective function for this problem is

$$\Pi_2\left(q_2, Q, q_1 | D_0 = d_0\right) = R\mathbb{E}_{t_o}\Big[\min\Big\{D_2, q_2 + \left(Q - d_0 - D_1\right)^+\Big\}\Big]$$
$$+S\mathbb{E}_{t_o}\big(q_2 + (Q - d_0 - D_1)^+ - D_2\big)^+ + s(q_1 - q_2) - c_2 q_2. \tag{5}$$

The expectation $\mathbb{E}_{t_o}$ in the above expression is conditional, reflecting that the distributions for random variables D_1 and D_2 are updated based on the observation $D_0 = d_0$. Next lemma characterizes the optimal value of q_2 at the end of observation period.

LEMMA 1. *The optimal number of components to assemble at $t = t_o$ is give by $q_2^*\left(Q, q_1, d_0\right) = \min\left\{\hat{q}_2\left(Q, d_0\right), q_1\right\}$, where $\hat{q}_2\left(Q, d_0\right)$ is the unconstrained maximizer of $\Pi_2\left(q_2, Q, q_1, | D_0 = d_0\right)$:*

$$\mathbb{P}_{t_o}\Big(D_2 \leq q_2 + \left(Q - d_0 - D_1\right)^+\Big) = \frac{R - c_2 - s}{R - S} \tag{6}$$

The left-hand side of (6) is the probability of having sufficient inventory to meet the demand during the time period $(t_o + l_2, t_o + L]$: When the firm does not stock-out at time $t_o + l$, and in such cases optimal value of q_2 together with previous replenishment Q should be greater than total demand $d_0 + D_1 + D_2$ to avoid stock-out during the period. On the other hand, when the firm stocks out at time $t_o + l$, and in such cases inventory at the arrival of q_2 unit is 0, and q_2 should be greater than demand D_2 to avoid stock-out. The right-hand side is the Newsvendor-type ratio with unit cost of under-stocking $R - c_2 - s$ and unit cost of over-stocking $c_2 + s - S$. The next lemma sheds light on the properties of this solution.

LEMMA 2. *(a) $q_2^*\left(Q, q_1, d_0\right)$ is, non-decreasing in q_1, non-increasing in Q, and non-decreasing in the realized value of observation period demand d_0.*

(b) Unconstrained optimal value $\hat{q}_2(Q, d_0)$ satisfies $\hat{q}_2(Q, d_0) \leq \min\left\{(z_1 - Q + d_0)^+, z_2\right\}$, where z_1 is the solution to $\mathbb{P}_{t_o}(D_1 + D_2 \leq z_1) = \frac{R - c_2 - s}{R - S}$ and z_2 is the solution to $\mathbb{P}_{t_o}(D_1 \leq z_2) = \frac{R - c_2 - s}{R - S}$.

We provide a brief explanation of the above properties. First consider part (a): Since $q_2^*\left(Q, q_1, d_0\right)$ is constrained by q_1, it follows that at a larger q_1 optimal $q_2^*\left(Q, q_1, d_0\right)$ would be at least as large as it is for smaller q_1. At a larger value of Q, inventory left-over before arrival of q_2 assembled pre-positioned components is (stochastically) larger, thus requiring a smaller optimal value of q_2.

Finally, a larger value of actual demand during observation period implies larger market acceptance of the product, and hence it requires a larger quantity q_2 to meet the demand. Part (b) of the lemma bounds unconstraint optimal by two quantities: First is $(z_1 - Q + d_0)^+$, which is the optimal quantity when $Q - d_0$ is large. In such cases there is inventory leftover from the first order Q, hence optimal quantity is determined by order-up-to level z_1. Second is z_2, which is the optimal quantity when $Q - d_0$ is small implying high likelihood of stockout prior to arrival of assembled components at $t = t_o + l_2$. This bound is easy to evaluate, and in our numerical study we show that it serves as a robust approximation to $\hat{q}_2(Q, d_0)$.

Lemma 2(a) implies that for each q_1 and Q, there exists a threshold value of observation period demand such that all pre-positioned components are assembled if d_0 is larger than the threshold value. Let $\bar{d}_0\left(Q, q_1\right) = \min\left\{d_0 : q_2^*\left(Q, q_1, d_0\right) = q_1\right\}$ denote such threshold value.

Given that the optimal value of q_2 is chosen according to Lemma 1, the expected profit at the time of placing initial orders as a function of Q and q_1 can be expressed as:

$$\Pi_1\left(Q, q_1\right) = R\mathbb{E}_0\left[\min\left\{D_0 + D_1, Q\right\}\right] - CQ + \mathbb{E}_0\left[\Pi_2\left(q_2^*\left(Q, q_1, D_0\right), D_0 \left| Q, q_1\right.\right)\right] - c_1 q_1. \tag{7}$$

Proposition 1. *The optimal values of q_1 is characterized by the following first order condition:*

$$\mathbb{P}_0\Big(D_2 \le q_1 + \left(Q - D_0 - D_1\right)^+, D_0 > \bar{d}\left(Q, q_1\right)\Big) = \frac{\left(R - c_2 - s\right)\mathbb{P}_0\big(D_0 > \bar{d}\left(Q, q_1\right)\big) - c_1 + s}{R - S}, \tag{8}$$

and the optimal value of Q is characterized by the following first order condition:

$$\mathbb{E}_0[\mathbb{P}_{t_o}(d_0 + D_1 + D_2 \le q_2^*\left(Q, q_1, d_0\right) + Q, d_0 + D_1 \le Q)] = \frac{R - C}{R - S}. \tag{9}$$

To explain (8), the optimality condition of q_1, consider a marginal unit of component being pre-positioned. This marginal unit will be assembled only if optimal value of q_2 hits the constraint $q_2 \le q_1$, which happens when $D_0 > \bar{d}_0\left(Q, q_1\right)$. The left-hand side captures how this additional unit directly influences the in-stock probability, conditional on its assembly and shipping to the retail store: if $d_0 + D_1 \le Q$, then the system is not stocked out at $t = t_o + l_2$ and $Q + q_1$ is matched with total demand $d_0 + D_1 + D_2$. On the other hand if $d_0 + D_1 > Q$, then the system is stocked-out at

$t = t_o + l_2$ and q_1 is matched with D_2. The right-hand side is the ~~Newsvendor~~ ratio of the unit under-stocking cost, and the sum of the under-stocking and unit over-stocking cost. In this case, the unit under-stocking cost $(R - c_2)\mathbb{P}_0\left(D_0 > \bar{d}_0(Q, q_1)\right) - c_1$ and the unit over-stocking cost is $c_2\mathbb{P}_0\left(D_0 > \bar{d}_0(Q, q_1)\right) + c_1$.

The explanation of (9), the optimality condition of Q, is similar and simpler. The left-hand side is how an additional unit influences in-stock probability. For a realized value of $D_0 = d_0$, this is determined by matching $d_0 + D_1$ with Q (for period $(0, t_o + l_2]$) and matching $d_0 + D_1 + D_2$ with $q_2^*(Q, q_1, d_0) + Q$ (for period $(t_o + l_2, t_o + L]$). Expectation with respect to D_0 is then taken to capture all realizations of D_0. In this case, the unit under-stocking cost is $R - C$ and the unit over-stocking cost is $C - S$, which gives the Newsvendor ratio of right-hand side.

Our next lemma provides the impact of pre-positioning strategy on the ordering decisions.

LEMMA 3. *The optimal ordering decisions under base case and with pre-positioning strategy satisfy* $Q^* \leq Q^b \leq Q^* + q_1^*$.

The above lemma illustrates that with pre-positioning strategy the initial commitment Q for finished units decreases. This is expected as pre-positioned components provide option for additional replenishment before the end of introduction phase. However, the total number of units (finished units *plus* components) ordered increases.

4.2. Minimum Order Quantities and Their Impact of Flexibility

In this section we consider pre-positioning strategy analyzed in the previous section together with component commonality. The value of component commonality derives from the fact that having common components allows firm the flexibility to change product assortment to suit the demand conditions. Minimum order quantity (MOQ) requirement hinders this flexibility to some extant. In order to illustrate the benefits of component commonality and the impact MOQ requirement has on this, we focus on a much simpler model that captures aforementioned trade-offs. We consider a two products, two stage model: In the first “procurement” stage the firm decides and orders specific and common components. In the second “assembly and fulfillment stage, the demand realizes and firm decides number of units of each product to assemble from specific and common components procured in the first stage. The second stage decision of assembly quantity of each product is constrained by the minimum order quantity.

Let $q_{1,j}$ denote the quantity of specific component for SKU j, and q_0 denote the quantity of common component procured in stage 1. And let $q_{2,j}$ denote the quantity of SKU j assembled in stage 2. We assume that both products have common unit revenue R, component specific component costs c_1 and assembly cost c_2. The unit cost of common component is c_0. Demand for SKU i is random number D_i, with realization denoted by d_i.

The second stage problem of optimal assembly decision can now be stated as following:

$$\Pi_2(q_{11}, q_{12}, q_0) = \max_{q_{21}, q_{22}} \left\{ \sum_{i=1,2} \left(R \min\{d_i, q_{2i}\} - c_2 q_{2i} \right) \right\}, \tag{10}$$

$$\text{s.t. } q_{21} + q_{22} \le q_0, \tag{11}$$

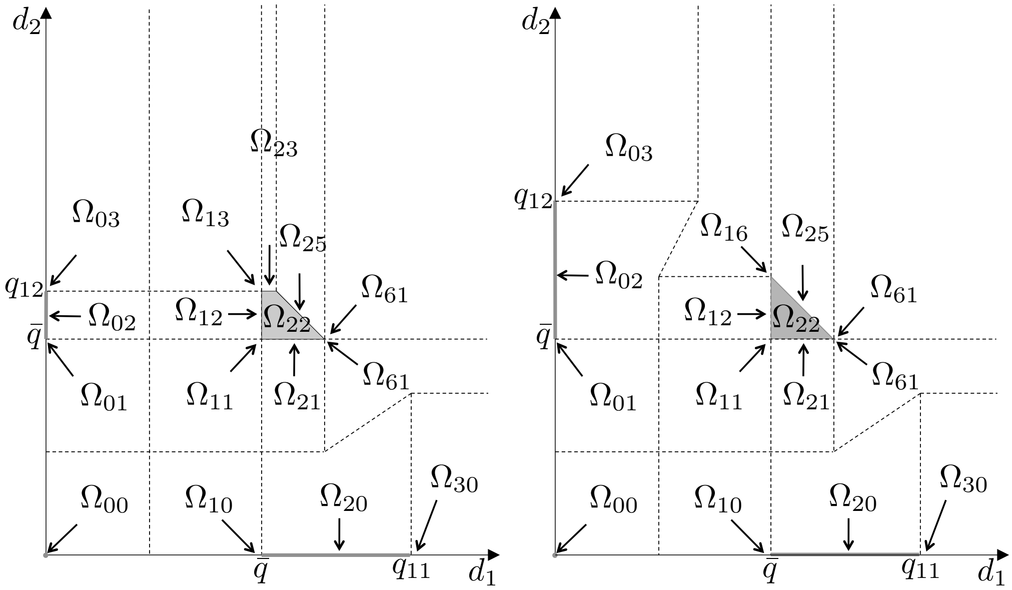
$$q_{2i} \le q_{1i} y_i, \qquad i \in \{1,2\}, \tag{12}$$

$$(q_{2i} - \bar{q}) y_i \ge 0, \qquad i \in \{1,2\}, \tag{13}$$

$$y_i \in \{0,1\}, \;\; q_i \ge 0, \qquad i \in \{1,2\}.$$

In the above formulation, binary variable y_i represents the decision of assembling SKU i. If $y_i = 1$, then constraints (12) and (13) imply that the quantity assembled q_{i2} must be smaller than procured specific components q_{1i}, and larger than minimum order quantity $\bar{q}$. If $y_j = 0$, then constraint (12) implies that $q_{j2} = 0$. Finally, the sum of assembled quantities of both SKUs must be no larger than the total number of common components q_0 procured in stage 1.

In the above formulation, note that if $q_{11} < \bar{q}$ or $q_{12} < \bar{q}$, the problem reduces to single SKU problem, and if $q_0 < \bar{q}$, the only feasible solution is $q_{21} = q_{22} = 0$. Further, at optimality $q_{12} \le q_0$ and $q_{11} \le q_0$, as assembly decisions q_{21} and q_{22} cannot exceed common component quantity q_0. To focus on nontrivial cases, we restrict our analysis to $\bar{q} \le q_{11} \le q_0$, $\bar{q} \le q_{12} \le q_0$ and $q_0 \ge \bar{q}$. Moreover when $q_0 < 2\bar{q}$, at-most one of the SKU can be assembled, and the problem simplifies to choosing which one of the two SKU's to assemble and in what quantity. In the next proposition, we solve



(a) For $q_{12} < q_0 - \bar{q}$ (b) For $q_0 - \bar{q} < q_{12} < q_0$

Figure 1 Optimal assembly decision for two SKUs under MOQ constraint

the optimal assembly problem for $q_0 \geq 2\bar{q}$, which represents the most complex case. We provide analysis for $\bar{q} \leq q_0 < 2\bar{q}$ in the online supplement. To simplify exposition, we assume that whenever multiple optima are present, the one with largest q_{12} is chosen (i.e., profit being the same, assembly of SKU 1 is prioritized).

Proposition 2. *When $q_0 \geq 2\bar{q}$, $q_{11} \geq \bar{q}$, $q_{12} \geq \bar{q}$, the optimal assembly decision is as follows: For $i, j \in \{1,2\}$ and $i \neq j$, let $\tilde{d}_i(d_j)$ denote the threshold value of d_i as function of d_j*

$$\tilde{d}_i(d_j) = \left((\min\{d_j, q_{1j}\} - q_0 + \bar{q})^+ + \frac{c_2}{R - c_2}\bar{q} \right) \frac{R - c_2}{R}.$$

1. *For $d_1 < \bar{q}$ and $d_2 < \bar{q}$, then $q_{2i} = 0$ if $d_i < \tilde{d}_i(d_j) = \frac{c_2}{R}\bar{q}$, and $q_{2i} = \bar{q}$, otherwise, for $i = 1, 2$.*
2. *For $d_1 < \bar{q}$ and $d_2 \geq \bar{q}$:*
   (a) *If $d_1 < \tilde{d}_1(d_2)$ then $q_{21} = 0$ and $q_{22} = \min\{d_2, q_{12}\}$.*
   (b) *If $\tilde{d}_1(d_2) \leq d_1 < \bar{q}$ then $q_{21} = \bar{q}$ and $q_{22} = \min\{d_2, q_{12}, q_0 - \bar{q}\}$.*
3. *For $d_1 \geq \bar{q}$ and $d_2 < \bar{q}$, the assembly decisions are symmetric to case 2 described above.*
4. *For $d_1 \geq \bar{q}$ and $d_2 \geq \bar{q}$, given assembly of SKU 1 is prioritize in case of multiple optimal allocations, $q_{21} = \min\{d_1, q_{11}, q_0 - \bar{q}\}$ and $q_{22} = \min\{d_2, q_{12}, q_0 - q_{21}\}$.*

We illustrate the optimal assembly decision given in Proposition 2 in Figure 1. In these plots, (d_1, d_2) space is partitioned into multiple regions, each representing a rule for assembly quantity decision: For $i = 1,2$ and $j = 3-i$, define the assembly quantity of SKU i as $a_i(0) = 0$, $a_i(1) = \bar{q}$, $a_i(2) = d_i$, $a_i(3) = q_{1i}$, $a_i(4) = q_0 - q_{1j}$, $a_i(5) = q_0 - d_j$ and $a_i(6) = q_0 - \bar{q}$. Then, region Ω_{kl} in Figure 1 corresponds to values of d_1 and d_2 such that $q_{21} = a_1(k)$ and $q_{22} = a_2(l)$. In each of these plots, the area marked with grey color represents feasible values of assembly quantity decisions. For each region, the arrow drawn in the region points towards the optimal assembly quantity decision for the region. In case of no arrow drawn in a region, the optimal assembly quantity decision at each point equals demand at that point.

Given the optimal assembly decision, the optimal procurement decision problem of the first stage is given by:

$$\Pi_1 = \max_{q_{11}, q_{12}, q_0} \left\{ \mathbb{E}\Pi_2(q_{11}, q_{12}, q_0) - c_1(q_{11} + q_{12}) - c_0 q_0 \right\},$$

where expectation in expression $\mathbb{E}\Pi_2(q_{11}, q_{12}, q_0)$ is taken over all realizations of D_1 and D_2. Note that the function $\mathbb{E}\Pi_2(q_{11}, q_{12}, q_0)$ is not continuous in q_{11}, q_{12} and q_0. For instance, for very small $\epsilon > 0$, the optimal assembly problem has different solutions for $q_0 = 2\bar{q} - \epsilon$ and $q_0 = 2\bar{q}$, resulting in discontinuity at $q_0 = 2\bar{q}$. Similar discontinuities exist at $q_0 = \bar{q}$, $q_{11} = \bar{q}$ and $q_{12} = \bar{q}$.

Proposition 3. (*a*) *The first-order condition for the optimal first stage procurement decisions of q_{11}, q_{21} and q_0 is given by:*

$$\begin{aligned}
\mathbb{P}(\Omega_{30}) + \mathbb{P}(\Omega_{31}) + \mathbb{P}(\Omega_{32}) &= \frac{c_1}{R - c_2}, \\
\mathbb{P}(\Omega_{03}) + \mathbb{P}(\Omega_{13}) + \mathbb{P}(\Omega_{23}) &= \frac{c_1}{R - c_2}, \\
\mathbb{P}(\Omega_{16}) + \mathbb{P}(\Omega_{25}) + \mathbb{P}(\Omega_{34}) + \mathbb{P}(\Omega_{61}) &= \frac{c_0}{R - c_2}.
\end{aligned}$$

(*b*) *The optimal number of specific components q_{11} is non-monotonous in q_0: It first decreases, and then increases in q_0.*

(*c*) *Optimal number of common components q_0 is non-monotonous in q_{11} and q_{12}.*

4.3. Linking Decisions in Introduction Phase with Stable Phase

In this section, we turn our focus towards linking the initial inventory decisions to inventory management during the stable phase of product life cycle. More specifically, we illustrate the impact of initial order quantities on inventory costs incurred after the end of introduction phase, and optimize initial ordering decisions that take into account this factor. In order to clearly highlight this link between the introduction phase and stable phase, we simplify the model by assuming that there is no systemic uncertainty in demand. Given this assumption, the uncertainty in demand prior to the launch is primarily due to the uncertainty in market acceptance of the product. In other words, after the product is launched the firm incurs demand at a deterministic rate μ for the product. Prior to the observation period the demand rate μ, which represents the level of market acceptance for the product, is an uncertain quantity with a known distribution.

As noted earlier, the initial inventory decisions affect inventory cost incurred after the end of introduction phase through the leftover inventory of initially ordered finished products and pre-positioned components. Therefore, to link the introduction phase with stable phase, we need to evaluate the cost implication of the finished product and component inventory that remains unsold/unused at the end of the introduction phase. Let x denote the inventory of finished products leftover at the end of introduction phase at time $t = t_o + L$, and y denote the inventory of components leftover after the (potential) assembly decision at $t = t_o$ for pre-positioned inventory. Then, for given values of x and y, and level of market acceptance μ, we develop an expression for terminal inventory cost $V(x, y, \mu)$. Note that this terminal inventory costs is a due of excess leftover inventory causing higher than optimal inventories after the end of introduction phase. The function $V(x, y, \mu)$ therefore depends on the inventory policy followed after the introduction phase, and how excess inventory is cleared if it is too large. For simplicity, we assume that after the introduction phase the firm follows a periodic review inventory model with review period length T. In other words, starting at time $t = t_o$ the firm places an order after every time interval of length T (equivalently, starting at time $t = t_o + L$, the firm receives a shipment after every time interval of length T). Further, for notational simplicity we assume that the firm continues to hold inventory of finished products and components until the demand eventually consumes them. Realistically, the firm would salvage inventory beyond a threshold value, and excess inventory would last only for limited number of periods. Our evaluation of $V(x,y,\mu)$ is therefore provides a lower bound to its value if salvage value is taken into account. As we illustrate in the electronic companion document, incorporating salvage value in $V(x,y,\mu)$, makes analysis tedious without significant insights.

Given periodic inventory decision with known demand process and no fixed costs, the optimal ordering policy (if and when it could be achieved) after the end of introduction phase is an order-up to policy. Further since the demand process has no systemic uncertainty, the optimal policy orders deterministic quantity μT to be delivered in every period of duration T and holds no safety inventory. In other words, in each period inventory is allowed to deplete to level 0 right before the arrival of next replenishment of size μT. The optimal inventory path during the stable phase therefore resembles a saw tooth diagram with inventory depleting from μT to 0 in each cycle. And, the desirable inventory level at the end of introduction phase (at $t = t_o + L$) is μT. If the value of inventory leftover x at the time $t = t_o + L$ from the pre-launch orders is smaller than μT, then at time t_o an order of size $\mu T - x$ is placed to achieve the optimal stable phase inventory profile. However, if $x > \mu T$ then attainment of the optimal stable phase inventory profile is delayed by some time, in other words for some time interval after the end of introduction phase the inventory levels are greater than optimal inventory levels. Similarly, the optimal component inventory in stable phase is 0. If $x + y \leq \mu T$, the component inventory y can be assembled immediately resulting in no additional component inventory after the end of introduction phase. However, if $x + y > \mu T$, then the firm will continue to hold component inventory for some time after the end of introduction phase, and will incur addition cost holding component inventory. The terminal cost function $V(x,y,\mu)$ measures cost of holding and clearing this excess inventory incurred after time $t = t_o + L$. Given these assumptions, the terminal cost function $V(x,y,\mu)$ is a continuous function of finished product inventory x and component inventory y which satisfies the following,

$$V(x,y,\mu) = -Cx - c_1 y + h_1 l_1 y, \quad \forall x \leq \mu T, \; x + y \leq \mu T, \tag{14}$$

$$\frac{\partial V(x,y,\mu)}{\partial x} = -C + HT\sum_{i=1}^{\infty}\mathbf{1}_{\left\{\mu\le\frac{x}{iT}\right\}} + h_1T\sum_{i=1}^{\infty}\mathbf{1}_{\left\{\frac{x}{iT}<\mu\le\frac{x+y}{iT}\right\}}, \tag{15}$$

$$\frac{\partial V(x,y,\mu)}{\partial y} = -c_1 + h_1l_1 + h_1T\sum_{i=1}^{\infty}\mathbf{1}_{\left\{\mu\le\frac{x+y}{iT}\right\}}. \tag{16}$$

In the above, (14) follows as all excess finished product and component inventory $x+y\le\mu T$ can be used towards meeting demand in the first inventory cycle following the end of the introduction phase, i.e., no additional holding cost would be incurred on these units. In addition, each of these leftover units would lead to not having to buy a unit in the stable phase, resulting in savings of unit cost then. However, since each component inventory has to wait for additional l_1 time after the observation period (i.e, from time $t=t_o$ to $t=t_o+L-l_2=t_o+l_1$) before it could assembled to be available as finished product at $t=t_o+L$, it incurs additional inventory holding cost of h_1l_1y. In (15): The first term $-C$ accounts to the saving incurred during the stable phase from a marginal leftover finished product; if $\mu\le x/iT$ then the marginal unit is held in inventory for at least iT periods incurring additional holding cost, described by the second term; the third term corresponds to the additional component holding cost incurred as marginal unit x will delay their assembly. Note that if $y=0$, the third term vanishes. Finally in (16), each additional unit of component leftover, results in: saving of cost $-c_1$ (not having to procure the component during stable phase); additional holding cost of h_1l_1 described earlier; and if $x+y\ge i\mu T$ then the component is held in inventory for at least i periods resulting in inventory holding cost described by the third term on the right-hand side. Finally, it follows from the expressions in (14), (15) and (16) that the function $V(x,y,\mu)$ is piecewise linear, with non-decreasing first derivatives with respect to x and y. It is therefore convex in x and y.

Now consider the base case strategy of our model. The expression for expected cost is:

$$\Pi^b = R\mathbb{E}_0\min\left\{\mu\left(t_o+L\right),Q\right\} - CQ - H\int_0^{t_o+L}\mathbb{E}_0\left(Q-\mu t\right)^+dt - \mathbb{E}_0V\left(Q-\mu\left(t_o+L\right),0,\mu\right),$$

where μ is a random number denoting the level of market acceptance for the product. Concavity of Π^b in Q follows directly from concavity of its various parts. The following result provides the optimal value of initial quantity that maximizes the profit incurred during the introduction phase.

Proposition 4. *The optimal order size for the initial ordering decision in base case is the unique solution Q to:*

$$\mathbb{P}_0\left(\mu \leq \frac{Q}{t_o+L}\right) = \frac{R-C-H\mathbb{E}_0\min\left\{t_o+L, \frac{Q}{\mu}\right\} - H\mathcal{T}_0(Q)}{R-C}, \tag{17}$$

where

$$\mathcal{T}_0(Q) = T\sum_{i=1}^{\infty}\mathbb{P}_0\left(\mu \leq \frac{Q}{t_o+L+iT}\right). \tag{18}$$

Right after the end of the introduction phase, i.e., at time $t = t_o + L$, the firm would like to achieve optimal steady-state inventory levels (i.e., inventory profile with saw-tooth pattern with inventory in each cycle going from μT to 0). This would not be achieved if finished product inventory at time $t = t_o + L$ is greater than μT, and in such cases the firm will hold more than optimal inventory level for some time after $t = t_o + L$. The duration of time by which the stable phase is delayed depends on Q, and function $\mathcal{T}_0(Q)$ given in (18) provides its expected value. The optimality condition in (17) is solution to a Newsvendor model adjusted for the impact of leftover inventory of the introduction phase on the stable phase.

Now consider the pre-positioning strategy: The decision to assemble pre-positioned components at time $t = t_o$ maximizes the following objective function:

$$\begin{aligned}\Pi_2(q_2,\mu|Q,q_1) = {} & R\min\left\{\mu(L-l_2), q_2 + (Q-\mu(t_o+l_2))^+\right\} - c_2q_2 \\ & -H\int_{t_o+l_2}^{t_o+L}\left(q_2 + (Q-\mu(t_o+l_2))^+ - \mu t\right)^+ dt \\ & -V\left(\left(q_2 + (Q-\mu(t_o+l_2))^+ - \mu l_1\right)^+, q_1 - q_2, \mu\right).\end{aligned}$$

Note that with no systemic uncertainty in the demand process, the value of μ is known at the time of this decision. The next proposition provides the optimal value of q_2.

Lemma 4. *Given initial order size Q of finished product, and q_1 of pre-positioned components, after the observation period t_o, the optimal number of pre-positioned components to assemble is given by $q_2^*(q_1, Q, \mu) = \min\left\{q_1, \mu l_1, (\mu(t_o+L) - Q)^+\right\}$.*

Since the cost rate of holding finished product inventory H is greater than that of holding component inventory h_1, it follows that in the absence of uncertainty, it is more cost effective to hold inventory in components rather than in finished products. This implies that it is optimal to assemble just enough components to meet the the demand incurred in $(t_o + l_2, L]$, which cannot be satisfied from the finished product inventory ordered earlier. When not constrained by q_1, this quantity is $\mu(t_o + L) - Q$ if no stock-out occurs until $t = t_o + l_2$, and is given by μl_1, otherwise. Since this quantity cannot be negative and is constrained by upper limit q_1, we get the optimal solution suggested in Lemma 4.

Folding the optimal solution $q_2^*(q_1, Q, \mu)$ back into the initial decision of ordering Q units of finished products and q_1 of components, we obtain the following objective function

$$\begin{aligned}\Pi_1(Q, q_1) = {} & R\mathbb{E}_0 \min\{\mu(t_o + l_2), Q\} - CQ - H \int_o^{t_o+l_2} \mathbb{E}_0 (Q - \mu t)^+ \, dt - c_1 q_1 \\ & + \mathbb{E}_0 \Pi_2(q_2^*(q_1, Q, \mu), \mu | Q, q_1).\end{aligned}$$

The next proposition provides the optimality conditions for initial ordering decisions Q and q_1.

Proposition 5. *Given initial order size Q of finished product, the optimal number of components to order for pre-positioning $q_1^*(Q)$is given by solution q_1 to*

$$\mathbb{P}_0\left(\mu \leq \max\left\{\frac{Q + q_1}{t_o + L}, \frac{q_1}{l_1}\right\}\right) = \frac{R - C - H\mathcal{T}_1(Q, q_1) - h_1 \tau_0(Q, q_1)}{R - C}, \tag{19}$$

and the optimal order size of finished product Q^* is given by solution Q to the following,

$$\mathbb{P}_0\left(\mu \leq \min\left\{\frac{Q + q_1^*(Q)}{t_o + L}, \frac{Q}{t_o + l_2}\right\}\right) = \frac{R - C - H(\mathcal{T}_2(Q, q_1^*(Q)) + \mathcal{T}_0(Q)) - h_1 \tau_1(Q, q_1^*(Q))}{R - C}, \tag{20}$$

where

$$\tau_0(Q, q_1) = l_1 \mathbb{P}_0\left(\mu \leq \max\left\{\frac{Q + q_1}{t_o + L}, \frac{q_1}{l_1}\right\}\right) + T \sum_{i=1}^{\infty} \mathbb{P}_0\left(\mu \leq \max\left\{\frac{q_1}{l_1 + iT}, \frac{Q + q_1}{t_o + L + iT}\right\}\right),$$

$$\mathcal{T}_1(Q, q_1) = \mathbb{E}_0\left[\left(\frac{q_1}{\mu} + \left(\frac{Q}{\mu} - (t_o + l_2)\right)^+\right) \mathbf{1}_{\left\{\frac{Q+q_1}{t_o+L} < \mu\right\}}\right],$$

$$
\begin{aligned}
\tau_1(Q,q_1) &= l_1 \mathbb{P}_0\left(\mu \le \min\left\{\frac{Q+q_1}{t_o+L}, \frac{Q}{t_o+l_2}\right\}\right) + T\sum_{i=1}^{\infty}\mathbb{P}_0\left(\mu \le \min\left\{\frac{Q+q_1}{t_o+L+iT}, \frac{Q}{t_o+l_2}\right\}\right) \\
&\quad -\tau_0(Q,0), \\
\mathcal{T}_2(Q,q_1) &= \mathbb{E}_0\left[\min\left\{\frac{Q}{\mu}, t_o+l_2\right\} + \left(\frac{Q+q_1}{\mu} - (t_o+l_2)\right)\mathbf{1}_{\left\{\frac{Q+q_1}{t_o+L}<\mu\le\frac{Q}{t_o+l_2}\right\}}\right] + l_1\mathbb{P}_0\left(\mu \le \frac{Q}{t_o+L}\right).
\end{aligned}
$$

Before explaining optimality conditions (19) and (20), we explain expressions $\tau_0(Q,q_1)$, $\mathcal{T}_1(Q,q_1)$, $\tau_1(Q,q_1)$ and $\mathcal{T}_2(Q,q_1)$. As noted earlier, once uncertainty in demand is resolved at the end of observation period, the optimal (desirable) inventory level of components is 0. When this cannot be achieved due to excess pre-positioned components (equivalently lower market acceptance μ of the product), the firm will continue to hold component inventory some time after $t = t_o$, and achieving optimal stable phase inventory would be delayed. Function $\tau_0(Q,q_1)$ provides the expected value of this delay in achieving desirable zero component inventory. In other words, a marginal prepositioned component spends $\tau_0(Q,q_1)$ time in inventory before being assembled into finished product unit. Next note that initially ordered finished product (ordering decision Q) also contributes to the aforementioned delay in achieving 0 component inventory level: A larger Q results in longer duration until more finished products are needed, hence each component waits for longer duration before it is assembled. Function $\tau_1(Q,q_1)$ captures this expected increase due to marginal unit of initially ordered finished product. Function $\mathcal{T}_1(Q,q_1)$ measures the expected time spent by a marginal unit of pre-positioned component during the introduction phase as a finished product. And function $\mathcal{T}_2(Q,q_1)$ measures the expected time spent by a marginal unit of initially ordered finished product during the introduction phase.

The optimality conditions in (19) and (20) can now be explained by comparing them to Newsvendor model. To explain the optimality condition of q_1 in (19), consider a marginal component being pre-positioned: This unit is sold during the introduction phase, only if all q_1 prepositioned units are assembled after observation period, which happens when $\mu(t_o+L) > Q+q_1$ and $\mu l_1 > q_1$. The probability on left-hand side of (19) is therefore that of having sufficient number of pre-positioned components, so that stock-out does not happen due to their unavailability. The marginal cost of under-stocking pre-positioned component is the profit earned on a marginal unit, which is given by unit margin $R-C$, minus cost of holding this unit until it is sold during the introduction phase $H\mathcal{T}_1(Q,q_1)$, minus additional holding cost imposed by this marginal unit after the end of introduction phase given by $h_1\tau_0(Q,q_1)$. The marginal cost of over-stocking pre-positioned component is the aforementioned holding cost incurred on a marginal unit.

To explain the optimality condition of Q in (20), consider a marginal finished product unit being ordered at $t=-L$. This unit will affect the probability of stockout if Q units are not sufficient to meet demand that is incurred until arrival for next replenishment at $t=t_o+l_2$, or if the maximum number of units $Q+q_1$ available during the introduction phase is not sufficient to meet the demand incurred until the end of introduction phase. The probability of the left-hand side in (20), is therefore the probability that stockout does not take place due to limited number of finished product units ordered. This marginal unit, in addition to earning the margin $R-C$, will have following cost impact: It incurs expected holding cost $H\mathcal{T}_2(Q,q_1)+H\mathcal{T}_0(Q)$ (former during the introduction phase and latter after the end of the introduction phase), and it delays assembly of component inventory resulting in additional component holding cost $h_1\tau_1(Q,q_1)$. As earlier, the marginal cost of over-stocking finished product consists of aforementioned inventory holding costs.

5. Implementation

In previous sections, we have described the issues associated with managing inventory during the early stages of product lifecycle in context of a cosmetics brand retailer based in Middle-East. An inventory management solution, which combines these key elements is developed and deployed for the firm which motivated this research. In this section, we outline additional challenges faced in implementing the solution and describe the impact of decisions made with its help.

Forecasting: One of key inputs for the solution is statistical description of demand. This includes distribution of demand as well as split of demand uncertainty into market specific uncertainty and product specific uncertainty. Five experts of the company individually did subjective monthly forecasts, which were used as basis for prior demand distributions roughly following the approach of Fisher et al (1999). The variance of this distribution is treated as total uncertainty in demand which combined both product and market specific uncertainties. This variance is then split between product specific variance and market specific variance in ratio 4:1. We experimented with different values of this ratio, and found 4:1 to be consistently very close to the worst-case scenario of uncertainty. Further, this closely echoes managements intuition that product specific uncertainty is the major part of the total uncertainty in during the initial phase. Econometric estimation of this spilt is remains an important question for future research.

Seasonality in Demand and Phased Launch: The firms demand is seasonal in nature during the month of August, high level of demand is spurred by religious festivals, whereas months (??) are slower. Furthermore, the firm decided to roll out the product line in phased manner, i.e., in some stores the product is launched 15 days ahead of all other stores. Both these factors create predictable variability in total demand over time. Monthly demand forecast, which ignores seasonality and phase launch, is scaled appropriately to arrive at seasonal and phase sensitive demand forecast over the introduction phase. All evaluations are carried out using this forecast.

Numerical Approximations: Given the complex nature of the problem after incorporating all the relevant factors, few simplifications are made in order to fasten the running time without considerable loss of accuracy. The exact expression for holding cost for a period $(t_1, t_2]$ starting with inventory y and demand $D_{(0,t]}$ is, $h\int_{t_1}^{t_2} \mathbb{E}\left(y - D_{(0,t]}\right)^+ dt$. This exact evaluation is very time consuming, specially since initial inventory y may be random number as well. In order to reduce this computational burden, we approximated this evaluation with $\frac{1}{2}h(t_2 - t_1)(\mathbb{E}(S - D_{(0,t_1)})^+ + \mathbb{E}(S - D_{(0,t_2)})^+)$. With component pre-positioning strategy, the decision on quantity to assemble after observation period is sufficiently complex, has to be evaluated several times. We approximated this decision with piecewise linear approximation discussed in **??**. This approximation is expected to work very well when there is high likelihood of large or little inventory at the arrival of current assembly order. We found that it works well in other cases too.

The problem of making assembly decision for multiple products with common component and minimum order quantity for assembly is significantly challenging. It can shown that a knapsack problem reduces to it, making this problem NP-hard. We developed the following approximation based on greedy heuristic for solving knapsack problem: For each SKU i, we sorted and assigned $\bar{q}_i$ common components in decreasing order of $\pi_i(\bar{q}_i) - \pi_i(0)$, where $\pi_i(q_i)$ is the expected profit from SKU i when q_i assembly is ordered after observation period. Note that some of the SKU's might be assigned no components. We reassign the left-over common components among the SKU's with $\bar{q}_i$ components. This assignment is done in decreasing order of gradient $\frac{\partial \pi_i(q_{0i})}{\partial q_{0i}}$. In ascending order of the final gradients, for each SKU i, q_{0i} is reassigned to 0 and these components are optimally reassigned among rest of the SKU's. If the change in total profit due to this is positive, then q_{0i} is permanently set to 0 or else permanently set to higher than $\bar{q}_i$. The gradient $\frac{\partial \pi_i(q_{0i})}{\partial q_{0i}}$ is replaced by $\frac{\pi_i(\bar{q}_i) - \pi_i(0)}{\bar{q}_i}$ when $q_{0i} = 0$. We used an approximation for computing $\frac{\partial \pi_i(q_{0i})}{\partial q_{0i}}$ and its inverse to make computation efficient. These approximations enable us to compute the expected profit for given quantity of finished goods, specific components and common components. To optimally find these quantities for single as well as multiple products we used Downhill simplex algorithm (Nelder, 1965)

The product line was launched in May 2009 over 100 retail stores. The product line comprised of 13 products and 147 SKUs (product variants). Based on the subjective forecast and parameter values, the quantity for each SKU is estimated. For ?? SKUs, manufacturing process was in two stages and allowed possibility of pre-positioning components and ?? products (?? SKUs) had common components allowing use of common component (combined with pre-positioning) strategy. The observation period is taken to be 1 month.

In order to access the impact of using over solution, we carried out retrospective analysis based on actual realized demand. In other words, we evaluated profit generate over all products if the quantity recommended by our solution was ordered.

Actual demand estimate was carried out using the sales data available from point of sale system, and inventory data imputed from sales and shipment data. The major hurdle in estimating actual demand for a month is due to the fact that actual demand is censored by availability of inventory, i.e., demand incurred during stock-out is lost. In order to estimate demand for a month, we assumed that demand rate during a month is stationary. Using the inventory data, we determined the duration of time in a month when product was in stock, and calculated the demand rate for the month. Demand estimate for the month is then estimated by multiplying demand rate and the duration of month. This approximation is reasonable when out-of-stock duration for each month is relatively small, which happens to be the case for most of products considered.

Table ?? illustrates our retrospective analysis for product ??. Here, we compare the order quantity actually placed by the firm, the base-case strategy of ordering only finished products, and the component pre-positioning strategy with common components.

6. Future Research and Concluding Remarks

Appendix

Proof of Lemma 2. Since $q_2^*(Q,q_1,d)=\min\{q_1,\hat{q}_2(Q,d)\}$, and $\hat{q}_2(Q,d)$ characterized by equation (6) is independent of q_1, it follows that $q_2^*(Q,q_1,d)$ is non-decreasing in q_1. This proves part (i). We prove part (iii) by showing that $\hat{q}_2(Q,d)$ has the same properties, the proof for part (ii) follows similarly. To prove that $\hat{q}_2(Q,d)$ is increasing in d, it is sufficient to show that the right-hand side of equation (6) decreasing in d. The right-hand side of equation (6) can be restated as:

$$\int_o^{Q-d}\left(\int_o^{Q+q_2-d-u} f_{D_1,D_2|D_0}(u,v|d)\,dv\right)du+\int_{Q-d}^{\infty}\left(\int_o^{q_2} f_{D_1,D_2|D_0}(u,v|d)\,dv\right)du,$$

where $f_{D_1,D_2|D_0}(u,v|d)$ is joint distribution of D_1 and D_2 conditional on realized value of $D_0=d$. Differentiating the right-hand side with respect to d results in,

$$\begin{aligned}&-\int_o^{q_2} f_{D_1,D_2|D_0}(Q-d,v|d)\,du+\int_o^{q_2} f_{D_1,D_2|D_0}(Q-d,v|d)\,du\\&\qquad-\int_o^{Q-d} f_{D_1,D_2|D_0}(u,Q+q_2-d-u|d)\,du\\&+\int_o^{Q-d}\left(\int_o^{Q+q_2-d-u}\frac{\partial}{\partial d}f_{D_1,D_2|D_0}(u,v|d)\,dv\right)du+\int_{Q-d}^{\infty}\left(\int_o^{q_2}\frac{\partial}{\partial d}f_{D_1,D_2}(u,v|d)\,dv\right)du.\end{aligned}$$

In the above first two terms cancel each other, the third term (without the minus sign) is positive and the last two terms are negative as D_1 and D_2 are stochastically increasing in d. Thus the right-hand side of equation (6) is decreasing in d. □

Proof of Proposition 1. Using the definition of $\bar{d}_0(Q,q_1)$, the profit function in equation can be restated as

$$R\mathbb{E}_0[\min\{D_0+D_1,Q\}]-CQ+\int_o^{\bar{d}_0(Q,q_1)}\Pi_2(\hat{q}_2(Q,d),d\,|Q,q_1)\,dF_0(d)$$

$$+\int_{\bar{d}_0(Q,q_1)}^{\infty} \Pi_2\left(q_1, d\,|Q, q_1\right) dF_0(d) - c_1 q_1.$$

The first dervative with respect to q_1 is,

$$= \int_{\bar{d}_0(Q,q_1)}^{\infty} \frac{d}{dq_1} \Pi_2\left(q_1, x\,|Q, q_1\right) dF_0(d) - c_1.$$

The second derivative is negative as well, implying concavity of the function in q_1. The first order condition therefore characterizes optimal value of q_1. From the expression of $\Pi_2\left(q_1, x\,|Q, q_1\right)$,

$$\begin{aligned}\frac{d}{dq_1} \Pi_2\left(q_1, d\,|Q, q_1\right) &= -R\mathbb{P}_{t_o}\left(d + D_1 + D_2 \leq q_1 + Q, d + D_1 \leq Q\right) \\ &\quad -R\mathbb{P}_{t_o}\left(D_2 \leq q_1 + Q, d + D_1 > Q\right) - R - c_2.\end{aligned}$$

Substituting this in the first derivative of $\Pi_1\left(Q, q_1\right)$ and equating to 0 results in the desired result. First derivative of $\Pi_1\left(Q, q_1\right)$ with respect to Q is

$$= R\left(1 - \mathbb{P}_0\left(D_0 + D_1 \leq Q\right)\right) - C + \frac{d}{dQ}\mathbb{E}_0\left[\Pi_2\left(q_2^*\left(Q, q_1, D_0\right), D_0\,|Q, q_1\right)\right].$$

Expanding the last term in the above expression and algebraic manipulation leads to

$$\begin{aligned}\frac{d}{dQ}\mathbb{E}_0\left[\Pi_2\left(q_2^*\left(Q, q_1, D_0\right), D_0\,|Q, q_1\right)\right] &= \mathbb{E}_0\left[R\mathbb{P}_{t_o}\left(d + D_1 + D_2 > q_2^*\left(Q, q_1, d\right) + Q, d + D_1 \leq Q\right)\right], \\ &= R\mathbb{P}_0\left(D_0 + D_1 \leq Q\right) \\ &\quad -\mathbb{E}_0\left[R\mathbb{P}_{t_o}\left(d + D_1 + D_2 \leq q_2^*\left(Q, q_1, d\right) + Q, d + D_1 \leq Q\right)\right].\end{aligned}$$

Substituting it back into the first derivative of $\Pi_1\left(Q, q_1\right)$ and equating to 0 leads to the desired result. □

Proof of Proposition 2. First consider $d_1 < \bar{q}$ and $d_2 < \bar{q}$ which is case 1 considered in the proposition. In such a case, $q_{2i} > \bar{q}$ cannot to be optimal, therefore only two choices $q_{2i} = 0$ and $q_{2i} = \bar{q}$ need to be considered. For $q_{2i} = \bar{q}$ to be optimal the following must be true,

$$R\min\{d_i, \bar{q}\} - c_2\bar{q} > 0,$$

or equivalently $d_i > \frac{c_2}{R}\bar{q}$. Since $q_0 \geq 2q_0$, decisions for q_{21} and q_{22} can be made independently.

Next consider $d_2 \geq \bar{q}$ and $d_1 < \bar{q}$ which is case 2 considered in the proposition (and is symmetric to case 3). Since $d_1 < \bar{q}$, $q_{21} > \bar{q}$ cannot be optimal. Therefore the choices for optimal assembly quantities are (i) $q_{21} = 0$ and $q_{22} = \min\{d_2, q_{12}, q_0\}$; and (ii) $q_{21} = \bar{q}$ and $q_{22} = \min\{d_2, q_{12}, q_0 - \bar{q}\}$. For the latter choice to be optimal, its profit must be larger, i.e.,

$$R\left(d_1 + \min\{d_2, q_{12}, q_0 - \bar{q}\}\right) - c_2\left(\bar{q} + \min\{d_2, q_{12}, q_0 - \bar{q}\}\right) \geq R\min\{d_2, q_{12}, q_0\} - c_2\min\{d_2, q_{12}, q_0\}.$$

On algebraic manipulation, the above can be rewritten as

$$d_1 \geq \frac{R-c_2}{R}\left((\min\{d_2,q_{12},q_0\}-q_0+\bar{q})^+ + \frac{c_2}{R-c_2}\bar{q}\right),$$

which is the desired expression for the threshold value $\tilde{d}_1(d_2)$. Note that for $d_2 < \bar{q}$, $\tilde{d}_1(d_2) = \dfrac{c_2}{R}\bar{q}$.

Finally consider case 4, in which $d_1 \geq \bar{q}$ and $d_2 \geq \bar{q}$. For this case the maximum achievable profit is no larger than the optimal profit without MOQ constraint. The latter is given by $(R-c_2)\left(\min\{d_1,q_{11}\}+\{d_2,q_{12},q_0-q_{21}\}\right)$, which is a achieved first satisfying demand d_1 to the fullest extant and then using remaining common components to meet demand for product 2, or by assembly decision $q_{21}=\min\{d_1,q_{11}\}$ and $q_{22}=\min\{d_2,q_{12},q_0-q_{21}\}$. If the proposed assembly decision $q_{21}=\min\{d_1,q_{11},q_0-\bar{q}\}$ and $q_{22}=\min\{d_2,q_{12},q_0-q_{21}\}$ achieves the same profit, then it is clearly optimal. If $\min\{d_1,q_{11}\}<q_0-\bar{q}$, then the proposed decision is equivalent to the optimal decision without MOQ constraint, hence the proposed decision is optimal. If $\min\{d_1,q_{11}\}>q_0-\bar{q}$, then proposed decision is $q_{21}=q_0-\bar{q}$ and $q_{22}=\bar{q}$, which results in maximum achievable profit $(R-c_2)q_0$, hence the proposed decision is optimal. This completes the proof for case 4, hence the proof of the proposition. □

PROOF OF PROPOSITION 3. Let $\pi_1(q_{11},q_{12},q_0)=\mathbb{E}\Pi_2\left(q_{11},q_{12},q_0\right)-c_1\left(q_{11}+q_{12}\right)-c_0q_0$, then it follows that $\pi_1(q_{11},q_{12},q_0)$ is continuous in q_{11}, q_{12} and q_0. (a) We show the first-order condition in q_0. Other first-order conditions follow similarly.

$$\frac{d\pi_1\left(q_{11},q_{12},q_0\right)}{dq_0}=\mathbb{E}\frac{d}{dq_0}\left(\Pi_2(q_{11},q_{12},q_0)-c_1(q_{11}+q_{12})-c_0q_0\right),$$

where change in order of integration (in form of expectation operation) and differentiation is allowed as function $\Pi_2(q_{11},q_{12},q_0)$ is continuous in q_0 (implying that derivative with respect of the limits of integration cancel out between different ranges of integration). From Proposition 2, we know that $\frac{d}{dq_0}\Pi_2(q_{11},q_{12},q_0)=R-c_2$ for $(d_1,d_2)\in\Omega_{16}\cup\Omega_{25}\cup\Omega_{34}\cup\Omega_{61}$, and $\frac{d}{dq_0}\Pi_2(q_{11},q_{12},q_0)=0$ elsewhere. This gives the requisite first order condition.

(b) It follows that $\pi_1(q_{11},q_{12},q_0)$ is continuous in q_{11}. Rewriting its first derivative as,

$$\frac{d\pi_1(q_{11},q_{12},q_0)}{dq_{11}}=\begin{cases}(R-c_2)\left(1-F_1(q_{11})\right)F_2\left(q_0-q_{11}\right)-c_1, & \text{for } q_{11}\leq q_0-\bar{q},\\ (R-c_2)\left(1-F_1(q_{11})\right)F_2\left(\tilde{d}_2(q_{11})\right)-c_1, & \text{for } q_{11}>q_0-\bar{q},\end{cases}$$

implies that $\pi_1(q_{11},q_{12},q_0)$ is differentiable everywhere except at point $q_{11}=q_0-\bar{q}$. Since $\pi_1(q_{11},q_{12},q_0)$ is continuous and takes small values at $q_{11}=0$ and $q_{11}\uparrow\infty$, it follows that the optimal value of q_{11} either satisfies the first-order condition or is at the point of the discontinuity of the first derivative $q_0 - \bar{q}$. The second derivative is

$$\frac{d^2\pi_1(q_{11}, q_{12}, q_0)}{dq_{11}^2} = \begin{cases} -(R-c_2)(1-F_1(q_{11}))F_2(q_0-q_{11})\left[\frac{f_1(q_{11})}{1-F_1(q_{11})} + \frac{f_2(q_0-q_{11})}{F_2(q_0-q_{11})}\right], & \text{for } q_{11} \le q_0 - \bar{q}, \\ (R-c_2)(1-F_1(q_{11}))F_2\left(\tilde{d}_2(q_{11})\right)\left[-\frac{f_1(q_{11})}{1-F_1(q_{11})} + \frac{R-c_2}{R}\frac{f_2\left(\tilde{d}_2(q_{11})\right)}{F_2\left(\tilde{d}_2(q_{11})\right)}\right], & \text{for } q_{11} > q_0 - \bar{q}, \end{cases}$$

In the above, the first expression for the second derivative is always negative, implying any local maxima satisfying $q_{11} \le q_0 - \bar{q}$ is unique in the interval. The sign of the second expression is determined by that of expression $-\frac{f_1(q_{11})}{1-F_1(q_{11})} + \frac{R-c_2}{R}\frac{f_2\left(\tilde{d}_2(q_{11})\right)}{F_2\left(\tilde{d}_2(q_{11})\right)}$, which is a decreasing function of q_{11}(since we assume increasing failure rate and $\tilde{d}_2(q_{11})$ is increasing in q_{11}). Thus id the second derivative is negative for some value of $q_{11} > q_0 - \bar{q}$, then it is negative for all greater values of q_{11}. This implies that if there exists a local maxima $q_{11} > q_0 - \bar{q}$ (at which first derivative is 0, and second is negative), then for all greater values of q_{11} both first and second derivative are negatives, hence there exists no other local maxima. Thus, there exists at most one local maxima $q_{11} \le q_0 - \bar{q}$, and at most one local maxima $q_{11} > q_0 - \bar{q}$.

Let q_{11}^* and $\hat{q}_{11}$ denote two local maxima satisfying

$$(R-c_2)(1-F_1(q_{11}^*))F_2(q_0-q_{11}^*) - c_1 = 0,$$

$$(R-c_2)(1-F_1(\hat{q}_{11}))F_2(\tilde{d}_2(\hat{q}_{11})) - c_1 = 0.$$

Since $q_0 - q_{11}$ is increasing in q_0 and $\tilde{d}_2(q_{11})$ is decreasing in q_0, it follows that if they exist, local maxima q_{11}^* is increasing in q_0, and $\hat{q}_{11}$ is decreasing in q_0. Further $q_0 - q_{11}^*$ is increasing in q_0 with $q_{11}^* > q_0 - \bar{q}$ for q_0 smaller than a threshold value, and $q_{11}^* < q_0 = \bar{q}$ for larger value of q_0.

If a local maxima $q_{11}^* < q_0 - \bar{q}$ exists: The point $q_{11} = q_0 - \bar{q}$ can be ruled out as local minima using earlier argument. For any $q_{11} > q_0 - \bar{q}$, we have

$$0 = (R-c_2)(1-F_1(q_{11}^*))F_2(q_0-q_{11}^*) - c_1 > (R-c_2)(1-F_1(q_{11}))F_2(\tilde{d}_2(q_{11})) - c_1,$$

which follows from nothing that $\tilde{d}_2(q_{11}) = \bar{q} - \frac{R-c_2}{R}(q_0 - q_{11}) < \bar{q} < q_0 - q_{11}^*$ and $q_{11}^* < q_{11}$ for such value of q_{11}. This implies that no other local maxima satisfying $q_{11} > q_0 - \bar{q}$ exists. Therefore if local maxima $q_{11}^* < q_0 - \bar{q}$ exists, it is unique and hence global maxima.

Thus we can conclude that for q_0 larger than a threshold value (such that $q_{11}^* < q_0 - \bar{q}$), the optimal value of q_{11} is q_{11}^*. For q_0 smaller than the threshold value, optimal q_{11} is either $q_{11} = q_0 - \bar{q}$ or $\hat{q}_{11}$. Comparing the objective function values at these two points

$$\pi_1(\hat{q}_{11}, q_{12}, q_0) - \pi_1(q_0 - \bar{q}, q_{12}, q_0) = \int_{q_0-\bar{q}}^{\hat{q}_{11}} \Big((R-c_2)\,(1-F_1(x))\, F_2\left(\tilde{d}_2(x)\right) - c_1 \Big)\, dx,$$

which on taking derivative with respect to q_0 gives the following:

$$\begin{aligned}\frac{d\left(\pi_1(\hat{q}_{11}, q_{12}, q_0) - \pi_1(q_0 - \bar{q}, q_{12}, q_0)\right)}{dq_0} = \quad & (R-c_2)\int_{q_0-\bar{q}}^{\hat{q}_{11}} (1-F_1(x))\, f_2\left(\tilde{d}_2(x)\right) \frac{d\tilde{d}_2(x)}{dq_0} dx \\ & + \Big((R-c_2)\,(1-F_1(\hat{q}_{11}))\, F_2\left(\tilde{d}_2(\hat{q}_{11})\right) - c_1\Big) \\ & - \Big((R-c_2)\,(1-F_1(q_0-\bar{q}))\, F_2\left(\tilde{d}_2(q_0-\bar{q})\right) - c_1\Big).\end{aligned}$$

In the right-hand side of above equation, (i) the first terms is negative, as $\frac{d\tilde{d}_2(x)}{dq_0} < 0$, (ii) the second term is 0, since $\hat{q}_{11}$ satisfies the first order condition, and (iii) the third term (without negative sign) is positive, since $q_0 - \bar{q} < \hat{q}_{11}$ and $\hat{q}_{11}$ satisfies the first order condition. Together these imply that $\pi_1(\hat{q}_{11}, q_{12}, q_0) - \pi_1(q_0 - \bar{q}, q_{12}, q_0)$ is decreasing, or there exist a threshold value below which $\hat{q}_{11}$ is optimal, and above which $q_0 - \bar{q}$ is optimal.

Thus we have shown that for small values of q_0 optimal q_{11} is $\hat{q}_{11}$ and is decreasing in q_0. Above a threshold value of q_0, optimal q_{11} is either $q_0 - \bar{q}$ or q_{11}^*, hence increasing in q_0. This completes the result.

(c) Without loos of generality, assume that $q_{12} < q_{11}$, then we have the following expression for the first derivative of π_1 with respect to q_0: For $q_0 \le q_{12} + \bar{q}$

$$\begin{aligned}\frac{\partial \pi_1(q_{11}, q_{12}, q_0)}{\partial q_0} =& (R-c_2)\Bigg[\int_{q_{12}}^{\infty}\int_{\tilde{d}_2(q_{12})}^{\infty} f_1(x_1) f_2(x_2) dx_2 dx_1 + \int_{q_0-\bar{q}}^{q_{12}}\int_{\tilde{d}_2(x_1)}^{\bar{q}} f_1(x_1) f_2(x_2) dx_2 dx_1 \\ & + \int_{\bar{q}}^{q_0-\bar{q}}\int_{q_0-x_1}^{\infty} f_1(x_1) f_2(x_2) dx_2 dx_1 + \int_{q_0-\bar{q}}^{\infty}\int_{\bar{q}}^{\infty} f_1(x_1) f_2(x_2) dx_2 dx_1 \\ & + \int_{q_0-\bar{q}}^{q_{11}}\int_{\tilde{d}_1(x_2)}^{\bar{q}} f_1(x_1) f_2(x_2) dx_1 dx_2 + \int_{q_{11}}^{\infty}\int_{\tilde{d}_1(q_{11})}^{\bar{q}} f_1(x_1) f_2(x_2) dx_1 dx_2\Bigg] - c_0,\end{aligned}$$

for $q_{12} + \bar{q} < q_0 \le q_{11} + \bar{q}$

$$\begin{aligned}\frac{\partial \pi_1(q_{11}, q_{12}, q_0)}{\partial q_0} =& (R-c_2)\Bigg[\int_{\bar{q}}^{q_{12}}\int_{q_0-x_1}^{\infty} f_1(x_1) f_2(x_2) dx_2 dx_1 + \int_{q_{12}}^{\infty}\int_{q_0-q_{12}}^{\infty} f_1(x_1) f_2(x_2) dx_2 dx_1 \\ & + \int_{q_0-\bar{q}}^{q_{11}}\int_{\tilde{d}_1(x_2)}^{\bar{q}} f_1(x_1) f_2(x_2) dx_1 dx_2 + \int_{q_{11}}^{\infty}\int_{\tilde{d}_1(q_{11})}^{\bar{q}} f_1(x_1) f_2(x_2) dx_1 dx_2\Bigg] - c_0,\end{aligned}$$

for $q_0 > q_{11} + \bar{q}$,

$$\frac{\partial \pi_1(q_{11}, q_{12}, q_0)}{\partial q_0} = (R-c_2)\left[\int_{q_0-q_{11}}^{q_{12}}\int_{q_0-x_1}^{\infty} f_1(x_1) f_2(x_2) dx_2 dx_1 + \int_{q_{12}}^{\infty}\int_{q_0-q_{12}}^{\infty} f_1(x_1) f_2(x_2) dx_2 dx_1\right] - c_0.$$

It follows from the above expressions that the first derivative is discontinuous at points $q_0 = q_{12} + \bar{q}$ and $q_0 = q_{11} + \bar{q}$, and continuous everywhere else. Thus, optimal value of q_0 is either a point that satisfies the first-order and second-order conditions, or one of the points of discontinuities. Let $w_1 \in (2\bar{q}, q_{12}+\bar{q}]$, $w_2 \in (q_{12}+\bar{q}, q_{11}+\bar{q}]$ and $w_3 \in (q_{11}+\bar{q}, \infty)$, then it follows from the expressions for first derivative that

$$\left.\frac{\partial \pi_1(q_{11}, q_{12}, q_0)}{\partial q_0}\right|_{q_0=w_1} > \left.\frac{\partial \pi_1(q_{11}, q_{12}, q_0)}{\partial q_0}\right|_{q_0=w_3},$$
$$\left.\frac{\partial \pi_1(q_{11}, q_{12}, q_0)}{\partial q_0}\right|_{q_0=w_2} > \left.\frac{\partial \pi_1(q_{11}, q_{12}, q_0)}{\partial q_0}\right|_{q_0=w_3}.$$

This together with the fact that the first derivative is in decreasing in q_0 for $q_0 > q_{11}+\bar{q}$, it follows that if $w_3 \in (q_{11}+\bar{q}, \infty)$ satisfies the first-order condition then it is unique global optima. Since $\frac{\partial^2 \pi_1}{\partial q_0 \partial q_{11}} > 0$ is this region, it follows that such a global optima increases with q_{11}. Further it can be shown that $\frac{\partial^2 \pi_1}{\partial q_0 \partial q_{11}} < -\frac{\partial^2 \pi_1}{\partial q_0^2}$, which implies that for such a global optima $q_0 + q_{11}$ is decreasing. Thus, as q_{11} increases, for some value $q_0 = q_{11}+\bar{q}$ would become optimal, and in such a case

$$\left.\frac{\partial \pi_1(q_{11}, q_{12}, q_0)}{\partial q_0}\right|_{q_0 \to q_{11}+\bar{q}-} > 0, \quad \text{and} \quad \left.\frac{\partial \pi_1(q_{11}, q_{12}, q_0)}{\partial q_0}\right|_{q_0 \to q_{11}+\bar{q}+} < 0.$$

Of these the first is decreasing in q_{11}, implying for some larger value of q_{11}, both derivatives in the above are negative, or optimal q_0 satisfies $q_0 < q_{11}+\bar{q}$. This optimal point is either $q_{11}+\bar{q}$ or a point in the interval $(2\bar{q}, q_{11}+\bar{q})$ that satisfies the first-order condition. In both the cases optimal q_0 is smaller than its earlier value. This proves that optimal q_0 is non-monotonous in q_{11}. $\square$